On Some Exponential Potentials for a Cosmological Scalar Field as Quintessence

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Abstract

We present general exact solutions for two classes of exponential potentials in a scalar field model for quintessence. The coupling is minimal and we consider only dust and scalar field. To some extent, it is possible to reproduce experimental results from supernovae.

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I. INTRODUCTION

In the last few years, new models of the universe have been built taking *dark energy* into account [1,2]. Together with baryons, cold dark matter, photons and neutrinos, a fifth component has been added, the socalled *quintessence* field Q [3–7] (or, in general, the *x-field* [8,9]). With respect to a more usual cosmological constant Λ-term, such a Q-field, even if it still implies a negative pressure contribution to the total pressure of the cosmic fluid, is characterized by the fact that its equation of state is given by $-1 < w_Q \equiv p_Q/\rho_Q < 0$, p_Q and ρ_Q being, respectively, the pressure and energy density of the Q-field. Actually, the interval $-1 < w_Q \lesssim -0.6$ is usually considered [10]. As a matter of fact, when $w_Q = -1$ we recover a constant Λ-term [11–13], which can be regarded as a measure of vacuum energy density, leading to the well known discrepancy between theory and observations [14,11,12], based on the question of why ρ_Q is so small with respect to typical particle physics scales. (But there are also mechanisms of relaxation of the cosmological constant during the initial inflationary stage, which could explain such a discrepancy; see [15], for instance.)

As well known, an interesting possibility to handle the presence of quintessence in the universe is to see it as given by a scalar field φ slowly rolling down its potential $V(\varphi)$. If we define

$$\rho_\varphi \equiv \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad ; \quad p_\varphi \equiv \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{1}$$

(dot indicating time derivative, and $V(\varphi)$ being the potential for φ), the slow rolling condition immediately gives $p_\varphi < 0$ and $w_\varphi \equiv p_\varphi/\rho_\varphi \simeq -1$. With such a negative pressure, the universe evolves like in a sort of present day *soft* inflationary scenario, so allowing to explain observations on supernovae [16–20] and why vacuum and matter densities are today comparable ('cosmic coincidence' problem [21,11,12]). (Alternatively to quintessence, a negative pressure and an explanation of current observations can also be obtained in a *Chaplygin cosmology* [22].)

Many cosmological models with a dynamical scalar field have been proposed, showing *scaling* solutions, i.e., such that at some time ρ_m and ρ_φ simultaneously depend on some powers n_1 and n_2 of the scale factor a, acting as attractors in the phase space. When $n_1 = n_2$, we have the socalled *self-tuning* solutions [23], which are typically driven by exponential potentials. These have been studied extensively [24–32], especially from a qualitative point of view (see [33] and references therein, for instance).

The simplest possibility is of course $V(\varphi) = \alpha e^{\lambda\varphi}$, which is often discarded (see discussion in [6,13]). More promising seems to be a combination of two terms $V(\varphi) = \alpha e^{\lambda\varphi} + \beta e^{-\lambda\varphi}$ [31].

In this paper, we consider a particular class of both these types from a different point of view, obtaining *general exact solutions*. This allows a very stringent comparison with experimental data on supernovae, so that also the first type seems to deserve further investigations; for the second type, we obtain a solution which can mimic very well the presence of a cosmological constant in the late evolution of the universe. Both of them are not, strictly speaking, *scaling solutions*, although this concept may be recovered in a more general sense.

Another experimental fact which we use is the strong evidence of a spatially flat universe [34]. Thus, we set the scalar curvature $k = 0$ in all our equations from the very beginning. However, as we shall see, the values of Ω_m and Ω_Λ ($\Omega\varphi$ in our case) derived from the experiments strongly depend on the model, so that some discussion is needed.

Mostly, the scalar field φ has been considered as minimally coupled to gravity, even if (more recently) a nonminimal coupling has also been introduced [35–39]. Here, we will consider a very simple model consisting of a two-component cosmological fluid: matter and scalar field. 'Matter' means baryonic + cold dark matter, with no pressure, and the scalar field is minimally coupled and noninteracting with matter. Clearly, this model cannot be used from the very beginning of the universe, but only since decoupling of radiation and dust. Thus, we do not take into account inflation, creation of matter, nucleosynthesis, etc. The main shortcut is that we cannot really check for the *tracker* feature [5,6] of the φ-solution. Anyway, as we shall see, there is no fine tuning of the initial conditions. (This part of the problem will be investigated better in a subsequent work.)

In Sec. II we take a particular exponential potential into account, deriving the general exact solution of the cosmological equations and, thereby, cosmological parameters as functions of time, so allowing the comparison with observational data. Sec. III is devoted to the same kind of considerations for a potential given by a linear combination of two such exponentials. Conclusions are drawn in Sec. IV.

II. AN EXPONENTIAL POTENTIAL

Let us consider a spatially flat, homogeneous and isotropic universe, filled with two noninteracting components only, i.e., pressureless matter (or *dust*) and a scalar field φ, minimally coupled to gravity. The cosmological equations are then

$$3H^2 = \frac{8\pi G}{c^2}(\rho_m + \rho_\varphi), \tag{2}$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3c^2}(\rho_m + \rho_\varphi + 3(p_m + p_\varphi)), \tag{3}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \tag{4}$$

where prime indicates derivative with respect to φ, $H \equiv \dot{a}/a$ is the Hubble parameter, $p_m = w_m\rho_m$ and $p_\varphi = w_\varphi\rho_\varphi$ are the equations of state for matter and scalar field. Let us stress that w_φ is *not constant*.

We set $w_m = 0$, so that $\rho_m = Da^{-3}$. The parameter $D \equiv \rho_{m0}{a_0}^3$ (the lower index '0' indicating present day values) is the amount of matter. The equations can also be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}(Da^{-3} + \frac{1}{2}\dot{\varphi}^2 + V(\varphi)), \tag{5}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right), \tag{6}$$

$$\ddot{\varphi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\varphi} + V'(\varphi) = 0. \tag{7}$$

In this Section, we consider the potential

$$V(\varphi) = B^2 e^{-\sigma\varphi}, \tag{8}$$

where B^2 is a generic positive constant and

$$\sigma^2 \equiv \frac{12\pi G}{c^2}. \tag{9}$$

(The minus sign in the exponential is irrelevant, since there is symmetry with respect to a change $\varphi \to -\varphi$.)

This type of potential leads to a late time attractor in a scalar-field dominated situation ($\Omega_\varphi = 1, w_\varphi = -0.5$) [13,31]. Being aware of such a behaviour, anyway, we stress that we are especially interested in the contemporary or, at most, the recent past regimes, where the situation is different. Usually, associated with an exponential potential, a scalar field is considered such that $\Omega_\varphi \equiv 8\pi G\rho_\varphi/(3c^2H^2)$ is practically constant during part of the matter-dominated era. This implies that assuming $w_\varphi \sim$ *constant* leads to a constant ratio of quintessence to matter energy density, so that Ω_φ (being ≤ 0.15 at the beginning of matter-dominated era, due to nucleosynthesis [28,29]) must remain small forever [6,13]. Mainly for such reasons, this kind of potential is not considered as suitable for a quintessence field.

The particular choice of Eq. (9) for σ allows for general exact integration of equations. Such a choice was in fact used in the context of inflationary theory by us [40,41] and others [42,43], with a scalar field only.

Let us concentrate on the second order equations (6) and (7), while Eq. (5), which is a first integral, is considered as a constraint on the integration constants. Let us introduce the new variables u and v, defined by the tranformation

$$a^3 = uv \quad ; \quad \varphi = -\frac{1}{\sigma}\log\frac{u}{v}, \tag{10}$$

which is always invertible (the Jacobian being $J = 2/\sigma$). We get for the potential

$$V(u,v) = B^2\frac{u}{v}, \tag{11}$$

and Eqs. (6) and (7) become

$$\ddot{u} = 0 \quad ; \quad \ddot{v} = \omega u, \tag{12}$$

where $\omega = \sigma^2 B^2 = 12\pi G B^2/c^2 > 0$. They are immediately integrated to

$$u(t) = u_1 t + u_2, \tag{13}$$
$$v(t) = \frac{1}{6}u_1\omega t^3 + \frac{1}{2}u_2\omega t^2 + v_1 t + v_2, \tag{14}$$

being u_1, u_2, v_1, and v_2 arbitrary integration constants. Taking into account Eq. (5), we find

$$\frac{2}{3}u_1 v_1 - \frac{1}{3}\omega u_2 - \frac{4\pi G}{c^2}D = 0. \tag{15}$$

Since D is a physical parameter, it would be natural to use it as *given* and derive one of the other constants. But this complicates calculations without substantial advantages, for D is not really known. Thus, we will determine D from Eq. (15)

$$D = \frac{c^2}{12\pi G}\left(2u_1 v_1 - \omega u_2\right), \tag{16}$$

which implies $2u_1 v_1 > \omega u_2$, so that u_1 and v_1 can never vanish and must have the same signs.

The well known [30,13] solution $\varphi = 2/\sqrt{3(1+\alpha)}\log t$, coming from the potential

$$V(\varphi) = \frac{2(1-\alpha)}{3(1+\alpha)^2}\exp\left(-\sqrt{3(1+\alpha)}\varphi\right), \tag{17}$$

is a very particular case of what we find in Eqs. (13) and (14). It is obtained by setting

$$B^2 = \frac{2(1-\alpha)}{3(1+\alpha)^2} \quad ; \quad \sigma = \sqrt{3(1+\alpha)} \quad ; \quad u_2 = v_2 = v_1 = 0. \tag{18}$$

Eq. (16) then gives $D = 0$, so that we get a model without matter, not really interesting in our context.

Without any special assumptions on constants, we can get many interesting quantities as functions of u and v (we do not write them explicitly in terms of t, for sake of brevity)

$$\rho_\varphi(u,v) = \frac{1}{\sigma^2}\left(\frac{(\dot{u}v - u\dot{v})^2}{2u^2v^2} + \omega\frac{u}{v}\right), \tag{19}$$

$$p_\varphi(u,v) = \frac{1}{\sigma^2}\left(\frac{(\dot{u}v - u\dot{v})^2}{2u^2v^2} - \omega\frac{u}{v}\right), \tag{20}$$

$$w_\varphi(u,v) = \frac{(\dot{u}v - u\dot{v})^2 - 2\omega u^3 v}{(\dot{u}v - u\dot{v})^2 + 2\omega u^3 v}, \tag{21}$$

$$H(u,v) = \frac{\dot{u}v + u\dot{v}}{3uv}, \tag{22}$$

$$\Omega_\varphi(u,v) = \frac{(\dot{u}v - u\dot{v})^2 + 2\omega u^3 v}{(\dot{u}v + u\dot{v})^2}, \tag{23}$$

$$\Omega_m(u,v) = \frac{24\pi G D uv}{c^2(\dot{u}v + u\dot{v})^2} = \frac{4u\dot{u}v\dot{v} - 2\omega u^3 v}{(\dot{u}v + u\dot{v})^2}, \tag{24}$$

the last equality coming from Eq. (16). It can be easily checked that $\Omega_m + \Omega_\varphi = 1$.

We now pass to simplify the situation with a suitable choice of initial conditions. Of course, the following choice is not the only one possible. A first check, anyhow, seems to indicate that nothing really important is lost, but this point will be investigated more carefully in the future.

If $t_{in} \equiv -u_2/u_1$ (which is always possible, being $u_1 \neq 0$), the scale factor a is zero, and we can show that there is no other time $t > t_{in}$ when this occurs again. We can thus fix the time origin in such a way that $a(0) = 0$. This condition has to be interpreted just as an arbitrary choice of the time origin. The real beginning (of physical meaning) for the model starts a little bit afterwards, at a time t_1. This delay is otherwise arbitrary, so that this setting does not seem to exclude important cases, as said before, and leads to a great simplification in the formulae. Now, $a(0) = 0$ implies $u_2 = 0$ or $v_2 = 0$, or both. If we set only one of them to zero, we obtain $\varphi(0) = \infty$ (which could be accepted, but is rather disturbing), and, most of all, $\Omega_\varphi(0) = 1$, which would mean an *initial* scalar-field dominated universe, with a neglectable content of other types of matter[1]. If we set $u_2 = v_2 = 0$ we get instead (in a matter-dominated situation, now)

$$\varphi(0) = -\frac{1}{\sigma}\log\frac{u_1}{v_1} \quad ; \quad \Omega_\varphi(0) = 0 \quad ; \quad D = \frac{c^2}{6\pi G}u_1 v_1. \tag{25}$$

We prefer to stick to this choice, so that we have

$$u(t) = u_1 t \quad ; \quad v(t) = \frac{1}{6}u_1\omega t^3 + v_1 t. \tag{26}$$

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[1]Of course, if we consider the general situation, the scalar field does dominate. But, as already mentioned, in our case we start after decoupling time, when a matter-dominated behaviour seems to be more natural.

Let us now define a time scale t_s such that $H(t_s) = 1/t_s$, which is of the order of the age of the universe. This leads to consider a dimensionless time $\tau \equiv t/t_s$. From Eqs. (22) and (26) we get

$$t_s^2 = \frac{6v_1}{\omega u_1}. \tag{27}$$

By means of these choices the formulae found above for the relevant cosmological parameters reduce to

$$\rho_\varphi = \frac{2(3+4\tau^2)}{\sigma^2 t_s^2 (1+\tau^2)^2}, \tag{28}$$

$$p_\varphi = -\frac{2(3+2\tau^2)}{\sigma^2 t_s^2 (1+\tau^2)^2}, \tag{29}$$

$$w_\varphi = -\frac{3+2\tau^2}{3+4\tau^2}, \tag{30}$$

$$a = (u_1 v_1 t_s^2 (1+\tau^2)\tau^2)^{1/3}, \tag{31}$$

$$(1+z)^3 = \frac{\tau_0^2(1+\tau_0^2)}{\tau^2(1+\tau^2)}, \tag{32}$$

$$H = \frac{2(1+2\tau^2)}{3t_s\tau(1+\tau^2)}, \tag{33}$$

$$\Omega_m = \frac{1+\tau^2}{(1+2\tau^2)^2}, \tag{34}$$

where $z \equiv a(\tau_0)/a(\tau) - 1$ is the redshift, and τ_0 indicates the present time.

If we define dimensionless pressure and energy density

$$\tilde{p}_\varphi \equiv \frac{\sigma^2 t_s^2}{2} p_\varphi \quad ; \quad \tilde{\rho}_\varphi \equiv \frac{\sigma^2 t_s^2}{2} \rho_\varphi, \tag{35}$$

we find the equation of state for the scalar field

$$\tilde{p}_\varphi = \tilde{\rho}_\varphi - 12 + 6\sqrt{4-\tilde{\rho}_\varphi}, \tag{36}$$

which is well approximated by

$$\tilde{p}_\varphi = -0.382\tilde{\rho}_\varphi - 0.196\tilde{\rho}_\varphi^2. \tag{37}$$

Eq. (36) is plotted in Fig. 1. In Fig. 2, we compare the plots of the two functions in Eqs. (36) and (37), and show that the approximation is quite good; comparison is also made with a straight line $\tilde{p}_\varphi = \langle w_\varphi \rangle \, \tilde{\rho}_\varphi$, where

$$\langle w_\varphi \rangle = \frac{1}{\varphi_0} \int_0^{\varphi_0} w_\varphi(\varphi) d\varphi \simeq -0.86, \tag{38}$$

being φ_0 the nowadays value of the scalar field (for its numerical value see below). As a matter of fact, we can see that the deviation we have, using the straight line instead of Eq. (36), is about 10%, with a maximum of 15% for high redshifts.

Also, using Eqs. (28) and (31), we derive the energy density ρ_φ as a function of the scale factor a. The plot of the logarithm of such a function versus $\log a$ is shown in Fig. 3 in comparison with $\log \rho_m$ versus $\log a$. It is thus clear that ρ_φ scales approximatively with an exponent $n \simeq -0.014$, in a first period, and (after a relatively short transition) is again scaling with a different exponent $n' \simeq -1.21$. We can say that we have here a generalization of the 'scaling ' concept.

We also see that, in the first period, matter is clearly dominating. Yet, $V(\varphi)$ is exponential. This seems to contraddict the derivation in [23]. Indeed, that result is obtained simply by dropping the energy contribution of the scalar field in the equations. May be that this is a too much crude approximation.

As a matter of fact, it is easy to see from Fig. 4 that w_φ is only very roughly constant. This makes irrelevant the usual arguments [6,13] against this type of potential, since (with our choice of initial conditions) it is also $\Omega_\varphi(0) = 0$, and $\Omega_\varphi(t_1)$ can be as small as we want. We have also to stress that $u_2 = v_2 = 0$ is not a fine tuning condition, because the real 'initial conditions' are more or less arbitrary and are situated at a preceding time as small as we like.

A last remark is that also the slow rolling condition is not fully verified. In fact the usual parameter x, which gives a measure of this condition, turns out to be

$$x \equiv \frac{1 + w_\varphi}{1 - w_\varphi} = \frac{1}{3} \frac{\tau^2}{1 + \tau^2}, \tag{39}$$

which is small only for very small τ, being $\simeq 0.13$ near the observable redshifts.

The main check for the solution in Eq. (26) is thus its capability to reproduce the experimental results, which we are going to do just below.

From Eq. (34) we get

$${\tau_0}^2 = \frac{1 - 4\Omega_{m0} + \sqrt{1 + 8\Omega_{m0}}}{8\Omega_{m0}}. \tag{40}$$

Once we give an acceptable value for Ω_{m0}, we obtain a value for τ_0. For instance, $\Omega_{m0} = 0.3$ gives $\tau_0 = 0.82$, and this implies $w_{\varphi 0} = -0.76$. If the value $H_0 = 100h\ km\ s^{-1} Mpc^{-1}$ is also given, we get

$$t_s = \frac{2(1 + 2\tau_0^2)}{3H_0\tau_0(1 + \tau_0^2)} = \frac{1.14}{H_0}; \tag{41}$$

assuming $h = 0.7$, we have $t_s = 15.8 \times 10^9$ years, and $t_0 = 13 \times 10^9$ years. It is also possible to obtain the relation between $w_{\varphi 0}$ and Ω_{m0}

$$w_{\varphi 0} = \frac{1 + 8\Omega_{m0} - 3\sqrt{1 + 8\Omega_{m0}}}{4(1 - \Omega_{m0})}, \tag{42}$$

which is plotted in Fig. 5. For $\Omega_{m0} = 0.2 \div 0.4$, we get $w_{\varphi 0} = -0.699 \div -0.811$, and the value -0.5 is reached only in the case of $\Omega_{m0} = 0$. It is also possible to obtain w_φ as a function of the redshift

$$w_\varphi = -\frac{2\zeta + \sqrt{\zeta(4 + \zeta)}}{\zeta + 2\sqrt{\zeta(4 + \zeta)}}, \tag{43}$$

where $\zeta \equiv (1+z)^3$. The plot is shown in Fig. 6. We can see that w_φ varies much and has values nonsmaller than -0.78, more or less. For $\zeta = 1$ (now) it is $w_\varphi = -0.77$, and already for $\zeta = 4$ ($z \simeq 0.59$) we find $w_\varphi \simeq -0.89$.

Another interesting quantity is the present value of φ. After staightforward algebra we get

$$\varphi_0 = -\frac{1}{\sigma} \log \left(\frac{27H_0^2}{16B^2} (1 - 4\Omega_{m0} + \sqrt{1 + 8\Omega_{m0}}) \right), \tag{44}$$

and we see that this value depends on the observed parameters and on the value of B^2, which has been before completely undetermined. Now, for $\tau_0 = 0.82$ and $t_0 = 13\times10^9$ years, we have $B^2 = 2.5\times10^{-47} \exp(-\sigma\varphi_0)\, GeV^4$. Considering $\varphi_0 \approx 1/6\ M_P$, M_P being the Planck mass, we see that $\exp(-\sigma\varphi_0) \approx 1$, and we can determine the unknown parameter for the potential

$$V(0) \equiv B^2 \approx 2.5\times10^{-47}\, GeV^4. \tag{45}$$

But we have also to observe that a 'little' change in φ_0 entails a 'large' change in B^2. For instance, if $\varphi_0 \approx M_P$, then $\exp(-\sigma\varphi_0) \approx 0.0025$ and B^2 changes of three orders of magnitude. Due to Eq. (25), we have that $\varphi(0) = -1/\sigma \log(2\pi G B^2 t_s^2/c^2)$. This means therefore that a relatively wide range of initial values of φ ends up to a much more narrow set of final φ_0's.

Thus, everything seems to work fine, but things are more complicated. Indeed, one has to ask what is really measured in the supernovae experiment. The value $\Omega_{m0} = 0.3$ is not a direct consequence of the data, since it depends on the model, which uses the constant Λ-term. What we really measure is the distance modulus, so that it is this quantity that we should compare in the two situations. Here, we limit ourselves to a very qualitative discussion.

Let us recall, then, the definitions of luminosity distance (in Mpc)

$$d_L = 3000(1+z) \int_0^z \frac{dz'}{H(z')}, \tag{46}$$

and distance modulus

$$\delta \equiv m - M = 5 \log_{10} d_L(z) + 25. \tag{47}$$

We have thus to compare this last quantity in the case when $H(z)$ is taken from the usual model with Λ [17,20], that is,

$$H(z) = H_0 \sqrt{(1+z)^2(1+\Omega_{m0}z) - z(2+z)(1-\Omega_{m0})}, \tag{48}$$

with the one obtained eliminating τ from Eqs. (32) and (33).

In Fig. 7 we compare δ with $\tilde{\delta}$ (let us mark with a $\sim$ the values for the model with Λ). The agreement is almost perfect, up to 0.06%. But there is a trick! $\tilde{\delta}$ was obtained from a value $\tilde{\Omega}_{m0} = 0.37$. Of course, this value is still in the possible range but at its limit. If we decide to trust strongly on the value $\tilde{\Omega}_{m0} = 0.3$ and want to obtain the same good agreement, we have to change the value of τ_0 to 1.22. This gives a very different value

$\Omega_{m0} = 0.16$ in the model with φ. This is again at the limit of possible estimates (due to other investigations on dark matter).

In conclusion, we see that this solution (with the potential in Eq. (8)) is indeed difficult to fully adapt to observed data, but for reasons which are not easy to investigate without general exact solutions. Moreover, it is not clearly incompatible (until we get better data); therefore, it seemed to us useful to present it in detail.

III. TWO EXPONENTIALS COMBINED

We now consider a combination of two exponentials, which will give us much better results, as expected. The procedure strictly follows the above one.

Let us consider the potential

$$V(\varphi) = A^2 e^{\sigma\varphi} + B^2 e^{-\sigma\varphi}, \tag{49}$$

with $\sigma^2 = 12\pi G/c^2$ as before, and A^2, B^2 arbitrary parameters. We use, now, the following change of variables

$$a^3 = \frac{u^2 - v^2}{4} \quad ; \quad \varphi = \frac{1}{\sigma}\log\frac{B(u+v)}{A(u-v)}, \tag{50}$$

which is invertible, provided that $a \neq 0$. This leads to

$$V(u,v) = 2AB\frac{u^2+v^2}{u^2-v^2}. \tag{51}$$

With these variables Eqs. (6) and (7) are rewritten as

$$\ddot{u} = \omega^2 u \quad ; \quad \ddot{v} = -\omega^2 v, \tag{52}$$

where now

$$\omega^2 = \frac{12\pi GAB}{c^2}. \tag{53}$$

Again, the integration is immediate, and gives the general solutions

$$u(t) = \alpha e^{\omega t} + \beta e^{-\omega t}, \tag{54}$$

$$v(t) = v_1 \sin(\omega t + v_2), \tag{55}$$

with α, β, v_1, v_2 arbitrary constants. As before, we derive D from the constraint in Eq. (5)

$$D = -\frac{c^2\omega^2(v_1 + 4\alpha\beta)}{24\pi G}. \tag{56}$$

Being $D > 0$, this implies $v_1 < -4\alpha\beta$. A change in the sign of v_1 has the only effect of changing the sign of φ, and interchanging A^2 with B^2. So, we can set $v_1 > 0$ without any loss of generality (the case $v_1 = 0$ is obviously equivalent to considering a Λ-term). As a consequence, α and β must be non zero and with opposite signs.

Again, we can write some important functions in terms of u, v

$$\rho_\varphi = \frac{2((\dot{v}u - \dot{u}v)^2 + \omega^2(u^4 - v^4))}{\sigma^2(u^2 - v^2)^2}, \tag{57}$$

$$p_\varphi = \frac{2((\dot{v}u - \dot{u}v)^2 - \omega^2(u^4 - v^4))}{\sigma^2(u^2 - v^2)^2}, \tag{58}$$

$$w_\varphi = \frac{(\dot{v}u - \dot{u}v)^2 - \omega^2(u^4 - v^4)}{(\dot{v}u - \dot{u}v)^2 + \omega^2(u^4 - v^4)}. \tag{59}$$

(This last expression gives $w_\varphi \simeq -1$ when $(\dot{v}u - \dot{u}v)^2 \ll u^4 - v^4$, which certainly can happen for sufficiently large times.)

For the redshift we have

$$(1+z)^3 = \frac{4{a_0}^3}{u^2 - v^2}, \tag{60}$$

and finally

$$H(u,v) = \frac{2(u\dot{u} - v\dot{v})}{3(u^2 - v^2)}, \tag{61}$$

$$\Omega_m(u,v) = -\frac{(u^2 - v^2)(\dot{v}^2 - \dot{u}^2 + \omega^2(u^2 + v^2))}{(u\dot{u} - v\dot{v})^2}, \tag{62}$$

$$\Omega_\varphi(u,v) = \frac{(\dot{v}u - \dot{u}v)^2 + \omega^2(u^4 - v^4)}{(u\dot{u} - v\dot{v})^2}, \tag{63}$$

with $\Omega_m + \Omega_\varphi = 1$, of course.

We now make a trial for the choice of the free parameters. We set again $a(0) = 0$, and ask for nonsingular $\varphi(0)$. The situation, and hence the interpretation, is the same as above. Thus, we pose $\alpha = -\beta = \lambda/2$, $v_2 = 0$. It is also possible to fix an arbitrary normalization for a and set $v_1 = 1$, obtaining at last

$$u(t) = \lambda \sinh(\omega t) \quad ; \quad v(t) = \sin(\omega t). \tag{64}$$

We get now

$$D = \frac{c^2\omega^2(\lambda^2 - 1)}{24\pi G}, \tag{65}$$

implying $|\lambda| > 1$, and

$$\varphi(0) = \frac{1}{\sigma} \log \frac{B(\lambda + 1)}{A(\lambda - 1)}. \tag{66}$$

We define a dimensionless time $\tau = \omega t$ and get

$$a(\tau) = \left(\frac{\lambda^2 \sinh^2 \tau - \sin^2 \tau}{4}\right)^{1/3}, \tag{67}$$

$$(1+z)^3 = \frac{\lambda^2 \sinh^2 \tau_0 - \sin^2 \tau_0}{\lambda^2 \sinh^2 \tau - \sin^2 \tau} \tag{68}$$

$$H(\tau) = \frac{\omega(\sin(2\tau) - \lambda^2 \sinh(2\tau))}{3(\sin^2 \tau - \lambda^2 \sinh^2 \tau)}, \tag{69}$$

$$w_\varphi(\tau) = \frac{\lambda^2(\cosh\tau \sin\tau - \cos\tau \sinh\tau)^2 + (\sin^4 \tau - \lambda^4 \sinh^4 \tau)}{\lambda^2(\cosh\tau \sin\tau - \cos\tau \sinh\tau)^2 - (\sin^4 \tau - \lambda^4 \sinh^4 \tau)}, \tag{70}$$

$$\Omega_m(\tau) = \frac{2(\lambda^2 - 1)(\cos(2\tau) + \lambda^2 \cosh(2\tau) - 1 - \lambda^2)}{(\sin(2\tau) - \lambda^2 \sinh(2\tau))^2}. \tag{71}$$

In comparison with the situation of Sec. 2, we now have one more free parameter. This gives the possibility of a much better agreement with observational data.

It is also possible to take into some account the final and initial values of φ, i.e., φ_0 and $\varphi_i = \varphi(0)$. If τ_0 is the present dimensionless time we get

$$\exp\left(\sigma(\varphi_i - \varphi_0)\right) = \frac{\lambda^2 \sinh \tau_0 + \lambda(\sin \tau_0 - \sinh \tau_0) - \sin \tau_0}{\lambda^2 \sinh \tau_0 - \lambda(\sin \tau_0 - \sinh \tau_0) - \sin \tau_0}. \tag{72}$$

When $\lambda >> 1$, we have $\varphi_i \simeq \varphi_0$. In fact, φ is practically constant and we have $w_\varphi \simeq -1$, with nearly perfect emulation of a cosmological constant. On the other hand, if $\lambda \simeq 1$, then $\exp\left(\sigma(\varphi_i - \varphi_0)\right) \simeq 0$. This can be interpreted as $\varphi_i \simeq \infty$, or, better, as the possibility of a wide range of φ_i's, with nearly the same final φ_0.

Whatever is λ, it is possible to obtain a good agreement with observational data. We give only two extreme cases, with $\lambda = 30$ and $\lambda = 1.1$.

In the first case (case I) we set

$$\lambda = 30 \quad ; \quad \tau_0 = 1.2 \quad ; \quad \omega = 2.8{\times}10^{-18} s^{-1} = 2.8{\times}10^{-42}\, GeV, \tag{73}$$

which gives

$$H_0 = 70 \quad ; \quad \Omega_{m0} = 0.3 \quad ; \quad w_{\varphi_0} = -0.999. \tag{74}$$

In the second case (case II) posing

$$\lambda = 1.1 \quad ; \quad \tau_0 = 0.44 \quad ; \quad \omega = 1.07{\times}10^{-18} s^{-1} = 1.07{\times}10^{-42}\, GeV \tag{75}$$

gives

$$H_0 = 70 \quad ; \quad \Omega_{m0} = 0.3 \quad ; \quad w_{\varphi_0} = -0.76. \tag{76}$$

In Figs. 8 and 9 the distance modulus δ for these two cases is compared with the Λ-term case. As before, the agreement is quite good, but now we have $\tilde{\Omega}_{m0} = 0.30$, so that, with this very rough analysis, it is impossible to make a distinction. It is also clear that, at least in this type of check, the parameter λ is degenerate.

Let us present again the plots of $\log(\rho)$ versus $\log(a)$. They are shown in Figs. 10 and 11. We see that the case I is practically indistinguishable from a Λ-term, while the case II is more similar to the situation of Sec. 2, with different scaling régimes. It is interesting to note that the régimes are now three, even if the last one seems to be important only in the remote future.

As a final result, we plot in Figs. 12 and 13 the equation of state for the scalar field. Now it is impossible to show an exact analytical expression, so that we only give the plots (in arbitrary units) in the two examined cases. It is interesting to note that in the case I, although so similar to the pure cosmological constant case, we nonetheless obtain a non trivial plot for the equation of state. But, clearly, this point deserves further investigation.

IV. CONCLUSIONS

We have discussed two particular kinds of potentials which have allowed the general exact integration of Friedmann equations in presence of dust (ordinary and cold dark matter) and scalar field. This has been achieved by performing suitable transformations of variables. Such transformations have not been guessed by chance, but resulted from a well known procedure, the *Nöther Symmetry approach* [40,41,44,39], based on an action principle. This was not said before because it was unnecessary to the main points of the discussion. Nevertheless, it now seems appropriate to stress the power of such a procedure, which allows to solve cosmological equations, often giving also informations on the potential and/or the possible coupling between the scalar field and the curvature of spacetime, without any limitation on the validity of the solution itself. (For details, see the literature quoted above, and the references therein.)

We have seen that, with a suitable choice of integration constants for both potentials, it is possible to reproduce the main recent results from supernovae (initially interpreted in a Λ-model), with considerable precision, especially in the case of the second potential. What is interesting, in our opinion, is that such kinds of models can bring to a different evaluation of an important quantity like Ω_{m0}. This, in a certain sense, sheds new light on the exponential potential of the first type, which is usually not considered as completely adequate for quintessence, for instance. But we have seen that, without considering *a priori* w_φ as a constant and having a general exact solution, something else can be learned.

Of course, all our discussion is still *qualitative*, in that we should need to make a more punctual analysis of observational data, and verify the best fit with the various models, in order to see if and when real differences arise. Anyway, our analysis already seems to confirm some of the considerations made in [45].

Another important point to note is that, to be realistic and cover the whole (or, at least, a substantially wider) range of the life of the universe, radiation (and hot dark matter) must be added into the game. This could allow to study the CMBR spectrum and the formation of structures. But it presumably destroys the possibility to integrate the system of the cosmological equations, leading to the necessity of using the results we established here only as a guide for a more complete analysis.

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